

# Constrained Codes as Networks of Relations

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**Abstract**—We revisit the well-known problem of determining the capacity of constrained systems. While the one-dimensional case is well understood, the capacity of two-dimensional systems is mostly unknown. When it is non-zero, except for the  $(1, \infty)$ -RLL system on the hexagonal lattice, there are no closed-form analytical solutions known. Furthermore, for the related problem of counting the exact number of constrained arrays of any given size, only exponential-time algorithms are known.

We present a novel approach to finding the exact capacity of two-dimensional constrained systems, as well as efficiently counting the exact number of constrained arrays of any given size. To that end, we borrow graph-theoretic tools originally developed for the field of statistical mechanics, tools for efficiently simulating quantum circuits, as well as tools from the theory of the spectral distribution of Toeplitz matrices.

## I. INTRODUCTION

While most storage devices record information on a two-dimensional surface, they emulate a one-dimensional environment by spacing tracks of recorded data. The distance between adjacent tracks in common devices is an order of magnitude larger than the distance between adjacent symbols along the track. The next big leap in storage density may be achieved by reducing the distance between tracks. This in turn, requires a two-dimensional constrained-coding scheme to be employed.

A two-dimensional constrained system  $\mathcal{S}_{n,m}$  is simply a set of  $n \times m$  arrays over some specified alphabet. The common example of such a system is the  $(d, k)$ -RLL constraint in which each row and each column of the array has runs of zeroes whose length is at least  $d$  and at most  $k$ . Other two-dimensional constraints forbid certain patterns in the arrays, such as the no-isolated-bit constraint in which every bit agrees with at least one of its four neighbors in the two-dimensional array.

An important measure associated with a constrained system is its *capacity*. Introduced by Shannon [14], the capacity of a constrained system  $\mathcal{S}$  is defined as

$$\text{cap}(\mathcal{S}) \stackrel{\text{def}}{=} \lim_{n,m \rightarrow \infty} \frac{\log_2 |S_{n,m}|}{nm}.$$

While the one-dimensional case is well understood, there is little known about the capacity of two-dimensional systems.

In the case of two-dimensional  $(d, k)$ -RLL systems, Ito et al. [8] characterized the values of  $(d, k)$  for which the capacity is zero. General bounds on the capacity of  $(d, k)$ -RLL were given by Kato and Zeger [11], constructive lower bounds for  $(d, \infty)$ -RLL by Halevy et al. [7], and non-constructive asymptotically-tight bounds for  $(0, k)$ -RLL by Schwartz and

Vardy [13]. For the specific case of  $(1, \infty)$ -RLL, Calkin and Wilf [3] gave a numerical estimation method using transfer matrices. Only for the  $(1, \infty)$ -RLL constraint on the hexagonal lattice, Baxter [1] gave an exact *but not rigorous*<sup>1</sup> analytical solution using the corner transfer matrix method.

Other two-dimensional constraints do not fare any better. Halevy et al. [7] considered bit stuffing encoders for the two-dimensional no-isolated-bit constraint to constructively estimate its capacity. Non-constructively, Forchhammer and Laursen [6], estimated this capacity using random fields.

The method we present is general enough to encompass a wide variety of constraints (both local and global) though its expressive power is yet undetermined. We use only *mathematically-rigorous* tools to obtain exact capacity solutions and polynomial-time algorithms. The method is based on a series of reductions: **1)** A constrained system is first reduced to a network of relations in a way which enables us to connect the number of satisfying assignments to the network with the number of constrained arrays. **2)** This network of relations is transformed to a weighted graph using holographic reductions in such a way that the number of satisfying assignments to the network equals the weighted perfect matching of the graph. This is a many-to-many reduction in which the individual perfect matchings do not correspond in any one-to-one way to the original satisfying assignments. **3)** Finally, the weighted perfect matching of the graph is expressed as a Pfaffian of a certain skew-symmetric matrix by using the FKT (Fisher-Kasteleyn-Temperley) method. The capacity of the original constrained system is the limit of the Pfaffian, while the Pfaffian itself provides a polynomial-time algorithm for counting the number of constrained arrays.

The reductions and algorithms are given very briefly due to the severe page limit, while the polynomial-time algorithm is not described at all. For the complete results and proofs the reader is encouraged to read [12]. In Section II we introduce holographic reductions and the FKT method. We apply these tools in Section III to an example constrained system. We conclude in Section IV with a description of further results.

## II. BACKGROUND

### A. Networks of Relations

We start by introducing networks of relations. For a discussion of the subject see [4] and references therein. Given some

<sup>1</sup>As Baxter notes in [2] page 409: “It is not mathematically rigorous, in that certain analyticity properties ... are assumed, and the results ... (which depend on assuming that various large-lattice limits can be interchanged) are used. However, I believe that these assumptions ... are in fact correct.”

ground set  $\Omega$ , a *relation on  $n$  variables* is a subset  $R \subset \Omega^n$ . A *network of relations* is a graph  $G = (V, E)$  where we associate with each vertex  $v \in V$  a relation  $R_v$  on  $\deg(v)$  variables being the ordered set of incident edges on  $v$ . We can now assign every edge a value from  $\Omega$  and check whether all the relations are satisfied. We say that an assignment is a *satisfying assignment* if for every  $v \in V$ , the relation  $R_v$  is satisfied.

**Example 1.** Let us take as an example the network of relations shown in Figure 1. We use the ground set  $\Omega = \{0, 1\}$  and define  $R_=_$  to be the all-equal relation on three variables,  $R_{\neq}$  to be the not-all-equal relation on three variables, and  $\phi_+$  to be the accept-all relation on one variable:

$x_1$	$x_2$	$x_3$	$R_=_$	$R_{\neq}$
0	0	0	1	0
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

$x_1$	$\phi_+$
0	1
1	1

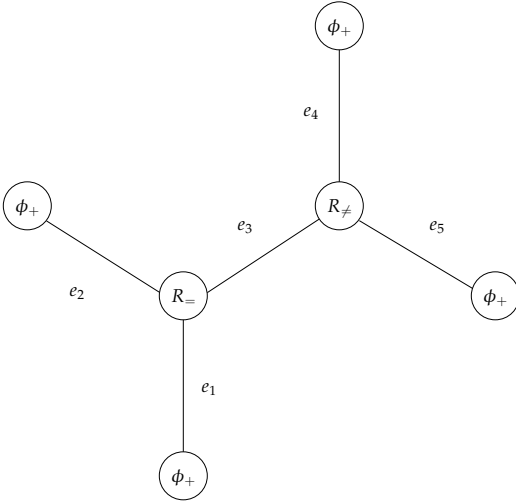


Figure 1. The network of relations of Example 1

We can easily see that there are exactly 6 distinct satisfying assignments to this network which we list below:

$$(e_1, \dots, e_5) \in \{(0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 1, 0, 1), (1, 1, 1, 1, 0)\}.$$

If we wanted to be completely accurate, we should have included a numbering of the incident edges to each vertex of Figure 1. However, since all the relations in this example are symmetric, this is unnecessary.  $\square$

### B. Holographic Reductions

Holographic reductions were introduced by Valiant in [18] to show certain counting problems may be solved in polynomial time, and in [17] to simulate quantum circuits efficiently.

Though the notion of networks of relations does not appear as such in his work, Valiant shows a many-to-many reduction from such networks to weighted graphs. This reduction preserves the total number of solutions, i.e., the number of satisfying assignments to the original network of relations equals the weighted perfect matching of the resulting graph. The reduction itself is realized by replacing each of the vertices of the original network with a small gadget.

Let  $G = (V, E)$  be a graph. A *perfect matching* is a subset of edges  $M \subseteq E$  such that every vertex  $v \in V$  is incident to exactly one of the edges in  $M$ . The set of all perfect matchings will be denoted  $\text{PM}(G)$ . We can now assign complex weights to the edges  $w : E \rightarrow \mathbb{C}$ , and define the *weighted perfect matching* of  $G$  to be

$$\text{PerfMatch}(G) \stackrel{\text{def}}{=} \sum_{M \in \text{PM}(G)} \prod_{e \in M} w(e).$$

Our aim is to replace vertices in the network of relations, with gadgets which somehow capture the original relations. The gadgets are called *matchgates* and the resulting graph is called a *matchgrid*. At this point, just like in [18], we require the graph  $G$  to be planar as well as all the matchgates we use, resulting in a planar matchgrid graph. This is perhaps the most restrictive requirement we face during the process. We are, however, able to use non-planar graphs, though at a cost of increased computational complexity. We omit the details of the holographic reduction and the intuition behind it.

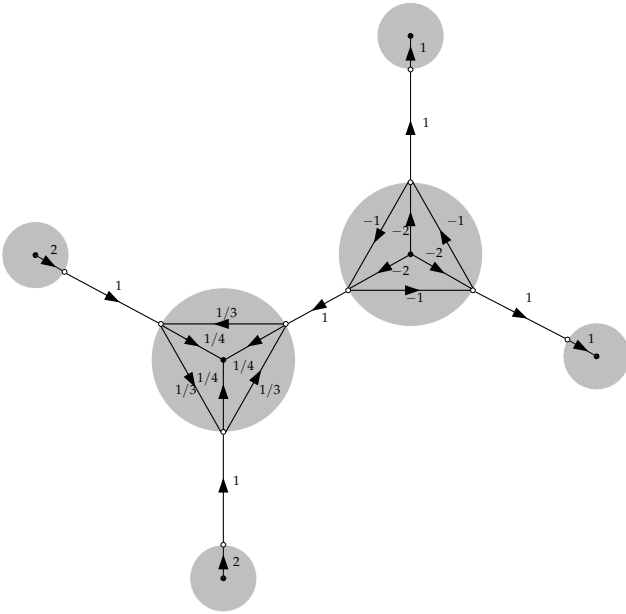
**Example 2.** We complete the matchgrid for the network of relations of Example 1. The resulting matchgrid is shown in Figure 2 (where the arrows on the edges are to be disregarded at the moment). Each gray circle represents the place of the original vertex in the network of relations which is now occupied with a matchgate subgraph gadget. The skeptical reader is encouraged to verify that the weighted perfect matching of this graph is indeed 6, as is the total number of satisfying assignments.  $\square$

### C. The FKT Method

The FKT method gives a simple expression for the weighted perfect matching of certain graphs which is also computable efficiently. It was developed independently and concurrently by Fisher and Temperley [15], [5], and by Kasteleyn [9].

Let  $G$  be a graph with weights on the edges, and let  $A = (a_{i,j})$  be its  $n \times n$  adjacency matrix where  $a_{i,j}$  is the weight of the edge between vertices  $i$  and  $j$ . Since we are interested in graphs with perfect matchings we assume  $n$  is even. An *orientation* of the an undirected graph  $G$  is simply an assignment of a direction to each of the edges of the graph. The solution given by Kasteleyn requires a special orientation called a *Pfaffian orientation* (for more details see [9]). Given a weighted graph  $G$ , and a Pfaffian orientation of its edges, we build a modified skew-symmetric adjacency matrix  $A = (a_{i,j})$  as follows:

$$a_{i,j} = \begin{cases} 0 & \text{no edge between } i \text{ and } j \\ w(e_{i,j}) & \text{if } i \rightarrow j \\ -w(e_{i,j}) & \text{if } j \rightarrow i \end{cases}$$



**Figure 2.** The complete matchgrid for the network of relations from Example 1 with a Pfaffian orientation of the edges

where  $i \rightarrow j$  denotes the edge between vertices  $i$  and  $j$  is oriented from  $i$  to  $j$ . Note that  $A$  is not the adjacency matrix of the graph  $G$  in the usual sense. Using this construction Kasteleyn [9] showed that

$$\text{PerfMatch}(G) = \pm \text{Pf}(A) = \pm \sqrt{\det(A)}$$

where,  $\text{Pf}(A)$  is the Pfaffian of  $A$  and  $\det(A)$  is the determinant of  $A$ . Since in most cases we know the sign of the outcome, the  $\pm$  may be easily fixed.

It now remains a matter of finding out which graphs allow a Pfaffian orientation. Such graphs are called *Pfaffian orientable*. In his later work, Kasteleyn [10] showed that all planar graphs are Pfaffian orientable, which is the reason we required matchgrids to be planar in the previous section. For planar graphs, it was shown in [10], that if we orient the edges such that every clockwise walk on a face of the graph has an odd number of edges agreeing, then that orientation is a Pfaffian orientation. As a result, a simple polynomial time algorithm which finds such an orientation is also shown. A Pfaffian orientation for Example 1 is shown in Figure 2.

### III. THE CAPACITY

Most constrained systems are easily defined by a finite set of forbidden patterns. For our example we choose a constraint we call the *Path-Cover Constraint (PC Constraint)*, and which we motivate by first examining its one-dimensional version. If we are given a graph  $G = (V, E)$ , a path-cover for the graph is a set of simple paths (open or closed) of positive length, which are vertex disjoint, and which cover all the vertices. An alternative way of stating this constraint is that given a graph, we assign either a 0 or a 1 to each of the edges, such that

when removing the edges with a 0, all the vertices remain with degree either 1 or 2.

In the one-dimensional case the graph  $G$  is simply the one-dimensional lattice with vertices  $V = \{v_0, v_1, \dots, v_n\}$  and edges  $E = \{(v_i, v_{i+1}) \mid 0 \leq i \leq n-1\}$ . It is easily seen that a valid PC assignment of values to edges is any assignment which does not contain two adjacent 0's. Thus, the one-dimensional PC constraint is the famous one-dimensional  $(0, 1)$ -RLL constraint (and with bit-flipping, the  $(1, \infty)$ -RLL constraint). The capacity in this case is known to be  $\log_2[(1 + \sqrt{5})/2] = 0.69424 \dots$

Turning to two-dimensions, we choose a the two-dimensional triangular grid as the graph  $G$ : we tile the plane with regular triangles, place a vertex at the center of each triangle, and draw an edge between vertices whose triangles share a face. Again, we assign either a 0 or a 1 to the edges of the graph such that after removing the edges assigned a 0, all the remaining vertices are of degree either 1 or 2.

There is a multitude of possible reductions of the constrained  $n \times n$  array on the plane, to a network of relations. We show a simple reduction for which the constrained arrays are in an “almost” one-to-one correspondence with the satisfying assignments to the network.

We think of the triangular grid as drawn on a plane. We replace each vertex of the grid with the relation  $R_{\neq}$  on three variables. This relation makes sure that the three adjacent cells do not contain the same bit, i.e., the forbidden pattern of PC. It is easy to be convinced that, if we ignore the perimeter of the array, every constrained array induces exactly one satisfying assignment and vice versa. The resulting network of relations is shown in Figure 3.

We do however have to take care of the perimeter of the array as well. To do so, we connect dangling edges to extra vertices of the accept-all relation  $\phi_+$ . Each such vertex has the potential of multiplying the number of satisfying assignments by a factor of 2. But since we have only  $O(n)$  such vertices, this does not change the capacity as calculated by counting the total number of satisfying assignments. The extra accept-all vertices are also shown in Figure 3.

It is easily seen that the network of relations we built is bipartite by noting that upright triangles are connected only to inverted triangles, and vice versa. Conveniently for us, for the main bulk of the network we have just one type of relation, but we do have to specify some as recognizers and some as generators. We arbitrarily choose to build a generator  $R_{\neq}$  in inverted triangles, and a recognizer  $R_{\neq}$  in upright triangles. For the perimeter of the network we need to implement  $\phi_+$  both as a generator and as a recognizer.

Finding a Pfaffian orientation for the graph is an easy task. The orientation is not necessarily unique. The extremely regular nature of the graph suggests the existence of a simple orientation. If we closely examine the network of relations in Figure 3, we see that, apart from the perimeter of the array, it is made up of a single *basic block* and its translations. The simplest basic block is just a recognizer  $R_{\neq}$  vertex and a generator  $R_{\neq}$  vertex. This basic block may be oriented



$p, k \geq 1$ , let an integrable  $p$ -variate function  $f : [-\pi, \pi]^p \rightarrow \mathbb{C}^{k \times k}$  and a multi-index  $n = (n_1, \dots, n_p)$ ,  $n_i \geq 1$  be given. The  $p$ -level Toeplitz matrix  $T_n(f)$  is defined as

$$T_n(f) \stackrel{\text{def}}{=} \sum_{j_1=-n_1+1}^{n_1-1} \dots \sum_{j_p=-n_p+1}^{n_p-1} J_{n_1}^{(j_1)} \otimes \dots \otimes J_{n_p}^{(j_p)} \otimes a_{j_1, \dots, j_p}(f)$$

where  $J_m^{(l)}$  denotes the matrix of order  $m$  whose  $i, j$  entry equals 1 if  $j - i = l$  and equals zero otherwise, and where

$$a_{j_1, \dots, j_p}(f) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^p} \int_{[-\pi, \pi]^p} f(x) e^{-i(j_1 x_1 + \dots + j_p x_p)} dx$$

is a matrix in  $\mathbb{C}^{k \times k}$  and  $\mathbf{i} = \sqrt{-1}$ .

**Theorem 3.** *If  $f : [-\pi, \pi]^p \rightarrow \mathbb{C}^{k \times k}$  is an integrable Hermitian matrix-valued function, then for any function  $F$ , uniformly continuous and bounded over  $\mathbb{R}$  it holds*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n_1 \dots n_p} \sum_{j_1=1}^{kn_1 \dots n_p} F[\lambda_j(T_n(f))] &= \\ &= \frac{1}{(2\pi)^p} \int_{[-\pi, \pi]^p} \sum_{j=1}^k F[\lambda_j(f(x))] dx \end{aligned}$$

where  $\lambda_j(M)$  denotes the  $j$ -th eigenvalue of  $M$ .

To apply this theorem to our needs we notice the following: First, for an  $n \times n$  matrix  $M$  we have  $\log_2 |\det(M)| = \sum_{i=1}^n \log_2 |\lambda_i(M)|$ . Second, we notice that  $A$  from (1) is a 2-level Toeplitz matrix, but it is skew-symmetric and not Hermitian as required. This is easily fixed by noting that  $\mathbf{i}A$  is Hermitian, and since the order of  $A$  is a multiple of 4, then  $\det(A) = \det(\mathbf{i}A)$ . Thus,  $A = T_n(f)$  where we define

$$\begin{aligned} f(\theta_1, \theta_2) &= \mathbf{i}[B + e^{\mathbf{i}\theta_1} \Delta_{6,2} - e^{-\mathbf{i}\theta_1} \Delta_{6,2}^T \\ &\quad + e^{\mathbf{i}\theta_2} \Delta_{7,3} - e^{-\mathbf{i}\theta_2} \Delta_{7,3}^T]. \end{aligned}$$

We can prove that the eigenvalues of  $\mathbf{i}A$  are bounded away from 0 (proof omitted). Then we can set  $F$  from Theorem 3 to behave like  $\log_2(|x|)$  on the closed interval containing the eigenvalues and still be bounded and uniformly continuous. Now we are done, and the capacity of the constraint is exactly calculated as

$$\begin{aligned} \text{cap}(S) &= \lim_{n \rightarrow \infty} \frac{\log_2 \sqrt{\det(A)}}{3n^2} \\ &= \lim_{n \rightarrow \infty} \frac{1}{6n^2} \log_2 |\det(T_n(f))| \\ &= \frac{1}{24\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log_2 |\det(f(\phi_1, \phi_2))| d\phi_1 d\phi_2 \\ &= \frac{1}{24\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log_2 |21 - 4 \cos \phi_1 - 4 \cos \phi_2 \\ &\quad - 4 \cos(\phi_1 - \phi_2)| d\phi_1 d\phi_2 \\ &= 0.72399217 \dots \end{aligned}$$

## IV. CONCLUSION

We presented a general method for calculating the exact capacity of two-dimensional constrained systems. Through a series of reductions, such systems are transformed to networks of relations, then to a limit of a Pfaffian, and by the theory of Toeplitz determinants, to a double integral. This method may be also adapted to provide, to our knowledge, the first polynomial time algorithm for counting the number constrained arrays of a given dimension. For more results the reader is referred to [12].

Many open questions remain regarding the parameters of the method which were not introduced in detail. The main question is that of the expressive power of this method and the systems which may be amenable to this kind of treatment.

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